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An Efficient Approximate Profiling Method for the Rack Gear Tool

Abstract: - This paper presents a simplified algorithm for the profiling of the rack gear tool to generate the active profile associated with rolling centrodes. The algorithm is based on the enveloping surfaces principles and reduces the computational problem by proposing an approximate representation of tool's profile using only 3 or 4 points on the profile to be generated by using Bezier representations of the surfaces. The algorithm computes the coefficients of a Bezier substitution polynomial instead of computing hundreds of points on the active profile. This simplifies the solution methodology and requires only a limited number of points along the profile to be generated. The algorithm is computationally efficient and can also be used in the case when the active profile to be generated is physically measured at a few points. Examples include mill tools, splined shafts and gears.

Key-Words: - Bezier substitution polynomial, tool profiling, approximate profiling method

1. Introduction

The vectorial or analytical representation of active surface profiles is widely used in engineering practice. Very often, these surfaces are approximately represented in forms that are suitable for numerical computations. For technological purposes, the surface representation of the active profiles is typically determined by accuracy needs in the manufacturing process. This is particularly the case for surfaces generated from enwrapping principles when typical representations include: description by poles; generation by surfaces enwrapping Litvin [1] and Radzevich [7]; based on the generating fundamentals theorems Olivier and Gohman [1, 7]; or on complementary theorems Oancea [5], [6] and Teodor [8].

The method proposed in this paper uses a representation of the profiles via a polynomial description as illustrated in Favrolles [3]. The polynomials are characterized primarily by the polynomial degree and the desire to describe the profile via a significantly smaller number of points on the conjugated profiles that would normally be feasible via classical methods. In addition, the representation errors on the conjugated profiles (both on the blank profile and on the tool profile) have to be bounded by an upper limit value that was predetermined according to the technological criteria of the generated profile.

Reference solutions are also constructed using previously published generation methods derived from surface enwrapping theorems. Comparisons between this new and these fundamental methods are performed to demonstrate the viability of the new method. Specific examples are illustrated for various profile shapes and generation kinematics. The algorithm determines the profile error for the generating profile with tools associated with rolling centrodes of elementary profiles on the workpiece.

2. Generation of a straight line profile using a rack-gear

Let's examine first the errors obtained in the rack-gear tool reciprocally enveloping with a straight line segment which belongs to a profile associated with a centrode with radius R_{rp} as shown in figure 1.

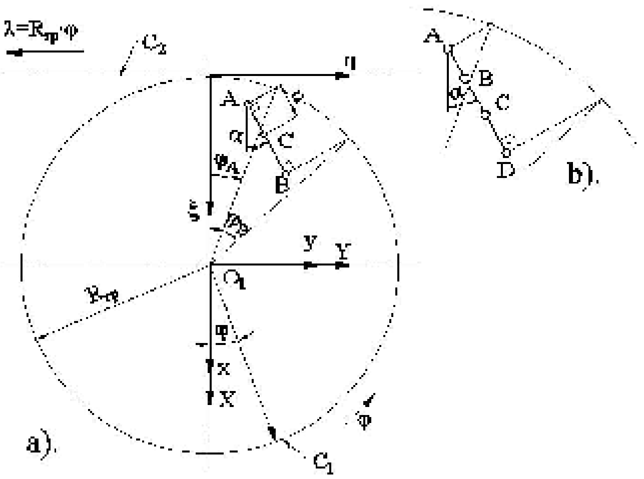

Fig. 1 – *Straight line profile AB (C_1, C_2, rolling centrodes); a) for 3 points-A,C,B; b) for 4 points-A,B,C,D.*

The following reference systems are used:

XY – the relative space associated with the blank;

xh – relative reference system joined with the rack-gear tool;

xy – global reference system.

The coordinates of the straight line ends, in the XY reference system are:

$$A[X_A, Y_A]; B[X_B, Y_B] \text{ and } \tan(a) = \frac{|Y_B - Y_A|}{|X_B - X_A|}. \quad (1)$$

The analytical representation of the profile is:

$$\Delta \left| \begin{aligned} X(u) &= X_A + u \cdot \cos a; \\ Y(u) &= Y_A + u \cdot \sin a; \end{aligned} \right. \quad (2)$$

with u a variable parameter; $u_{\min} = 0$ and $u_{\max} = \sqrt{(X_A - X_B)^2 + (Y_A - Y_B)^2}$.

The relative motion of the XY and xh reference systems may be described by

$$\begin{bmatrix} x \\ h \end{bmatrix} = \begin{bmatrix} \cos j & -\sin j \\ \sin j & \cos j \end{bmatrix} \cdot \begin{bmatrix} X_A + u \cdot \cos a \\ Y_A + u \cdot \sin a \end{bmatrix} - \begin{bmatrix} -R_{rp} \\ -R_{rp} \cdot j \end{bmatrix}, (3)$$

to generate the profile family:

$$(\Delta)_j \left| \begin{aligned} x &= X(u)\cos j - Y(u)\sin j + R_{rp}; \\ h &= X(u)\sin j + Y(u)\cos j + R_{rp} \cdot j. \end{aligned} \right. \quad (4)$$

The envelope of this family represents the rack-gear tool's profile in the rolling movement of the two centrodes.

The parametric equations (4) allow for the determination of the coordinates of points belonging to the rack-gear tool's profile by varying the u parameter between given bounds.

For $u = 0$, the coordinates of point A on the rack-gear tool profile $[x_A, h_A]$ are:

$$\begin{aligned} x_A &= X_A \cos j_A - Y_A \sin j_A + R_{rp}; \\ h_A &= X_A \sin j_A + Y_A \cos j_A + R_{rp} \cdot j_A, \end{aligned} \quad (5)$$

where j_A is the value of the j parameter corresponding to point A on the profile to be generated.

Similarly, for $u = u_{\max}$ the coordinates on the rack-gear profile corresponding to point B $[x_B, h_B]$ are:

$$\begin{aligned} x_B &= X_B \cos j_B - Y_B \sin j_B + R_{rp}; \\ h_B &= X_B \sin j_B + Y_B \cos j_B + R_{rp} \cdot j_B, \end{aligned} \quad (6)$$

where j_B is the value of the parameter j corresponding to point B on the profile to be generated.

According to the Willis theorem [1] at the ends of the segment Δ corresponding to the parameter values, j_A and j_B, the normal to the profile must intersect the associated centrode. Thus, knowing the normal to the profile to be generated as

$$[X - X(u)]X'_u + [Y - Y(u)]Y'_u = 0 \qquad (7)$$

and the circular centrode equations:

$$X = R_{rp} \cdot \cos j\,;\ Y = R_{rp} \cdot \sin j\,, \qquad (8)$$

allow for the determination of the values of the angles corresponding to the characteristic points of the *AB* segment:

$$j_A = \arccos\left(\frac{X_A \cos a + Y_A \sin a}{R_{rp}}\right) + a \qquad (9)$$

and $$j_B = \arccos\left(\frac{X_B \cos a + Y_B \sin a}{R_{rp}}\right) + a\,. \qquad (10)$$

Similarly, one can define the corresponding angle for other points on the profile to be generated. For a particular choice of $u = 0.5 \cdot u_{\max}$, at point C:

$$\begin{cases} X_C = 0.5 \cdot X_A + 0.5 \cdot X_B; \\ Y_C = 0.5 \cdot Y_A + 0.5 \cdot Y_B. \end{cases} \qquad (11)$$

And so the coordinates of point belonging to the rack-gear profile are:

$$\begin{aligned} x_C &= X_C \cos j_C - Y_C \sin j_C + R_{rp}; \\ h_C &= X_C \sin j_C + Y_C \cos j_C + R_{rp} \cdot j_C \end{aligned} \qquad (12)$$

and $$j_C = \arccos\left(\frac{X_C \cos a + Y_C \sin a}{R_{rp}}\right) + a\,. \qquad (13)$$

The *three-point* assembly, *A*, *C* and *B*, determine the profile (5) expressed in the following tabular form:

u dimensional parameter	Work Piece profile		Enveloping condition	l dimensionless parameter	Tool profile	
	X	Y			x	h_A
$u = 0$	X_A	Y_A	see (9)	$l = 0$	x_A	h_A
$u = \left(\frac{1}{2}\right) \cdot u_{\max}$	X_C	Y_C	see (13)	$l = \frac{1}{2}$	x_C	h_C
$u = u_{\max}$	X_B	Y_B	see (10)	$l = 1$	x_B	h_B

(14)

From the representation (14), one can construct an approximated polynomial representation of the tool's profile using a second order polynomial:

$$\begin{aligned} x &= l^2 \cdot A_x + 2l(1-l) \cdot C_x + (1-l)^2 \cdot B_x; \\ h &= l^2 \cdot A_h + 2l(1-l) \cdot C_h + (1-l)^2 \cdot B_h, \end{aligned} \qquad (15)$$

where $A_x, A_h, C_x, C_h, B_x, B_h$, are polynomial coefficients computed as illustrated in table 1, from the equations system defined from (15), for various values of l parameter (0; 0.5; 1).

Table 1. Straight line segment, 2nd degree approximation polynomial coefficients identification

u	**Primary profile**	**Enwrapping condition**
0	X_A, Y_A	$j_A = \arccos\left(\frac{X_A \cos a + Y_A \sin a}{R_{rp}}\right) + a$
0.5 $u_{\max}$	$X_C = 0.5 \cdot X_A + 0.5 \cdot X_B$ $Y_C = 0.5 \cdot Y_A + 0.5 \cdot Y_B$	$j_C = \arccos\left(\frac{X_C \cos a + Y_C \sin a}{R_{rp}}\right) + a$
$u_{\max}$	X_B, Y_B	$j_B = \arccos\left(\frac{X_B \cos a + Y_B \sin a}{R_{rp}}\right) + a$
λ	**Points on rack-gear profile**	**Approximation polynomial coefficients**
1	$x_A = X_A \cos j_A - Y_A \sin j_A + R_{rp}$ $h_A = X_A \sin j_A + Y_A \cos j_A + R_{rp} \cdot j_A$	$A_x = x_A$ $A_h = h_A$
0.5	$x_C = X_C \cos j_C - Y_C \sin j_C + R_{rp}$ $h_C = X_C \sin j_C + Y_C \cos j_C + R_{rp} \cdot j_C$	$C_x = (x_C - 0.25 \cdot x_A - 0.25 \cdot x_B) \cdot (1/0.5)$ $C_h = (h_C - 0.25 \cdot h_A - 0.25 \cdot h_B) \cdot (1/0.5)$
0	$x_B = X_B \cos j_B - Y_B \sin j_B + R_{rp}$ $h_B = X_B \sin j_B + Y_B \cos j_B + R_{rp} \cdot j_B$	$B_x = x_B$ $B_h = h_B$

When the points on the initial profile are known by direct physical measuring, the parameter λ may be calculated with the simple relation

$$\lambda = \left|\overline{AC}\right| \big/ \left|\overline{AB}\right| . \qquad (16)$$

Similarly, if a 3rd degree polynomial is used, one can write:

$$\begin{aligned} x &= \lambda^3 \cdot A_x + 3\lambda^2(1-\lambda)\cdot B_x + \\ &+3\lambda(1-\lambda)^2\cdot C_x + (1-\lambda)^3\cdot D_x; \\ \eta &= \lambda^3 \cdot A_\eta + 3\lambda^2(1-\lambda)\cdot B_\eta + \\ &+3\lambda(1-\lambda)^2\cdot C_\eta + (1-\lambda)^3\cdot D_\eta, \end{aligned} \qquad (17)$$

where the coefficients $A_x, A_\eta, B_x, B_\eta, C_x, C_\eta, D_x, D_\eta$ are computed as shown in table 2, from the equations system defined from (17), for various values of λ parameter (0; 0.33; 0.66; 1). Note that for the intermediate points on the active profile other solutions are possible for an approximate representation of the active profile.

Table 2. Straight line segment, 3rd degree approximation polynomial coefficients identification

u	**Primary profile**	**Enwrapping condition**
0	X_A, Y_A	$\varphi_A = \arccos\left(\dfrac{X_A\cos\alpha + Y_A\sin\alpha}{R_{rp}}\right) + \alpha$
1/3 u_{max}	$X_B = X_A + \frac{1}{3}(X_B - X_A)$ $Y_B = Y_A + \frac{1}{3}(Y_B - Y_A)$	$\varphi_B = \arccos\left(\dfrac{X_B\cos\alpha + Y_B\sin\alpha}{R_{rp}}\right) + \alpha$
2/3 u_{max}	$X_C = X_A + \frac{2}{3}(X_B - X_A)$ $Y_C = Y_A + \frac{2}{3}(Y_B - Y_A)$	$\varphi_C = \arccos\left(\dfrac{X_C\cos\alpha + Y_C\sin\alpha}{R_{rp}}\right) + \alpha$
u_{max}	X_D, Y_D	$\varphi_D = \arccos\left(\dfrac{X_D\cos\alpha + Y_D\sin\alpha}{R_{rp}}\right) + \alpha$
λ	**Points on rack-gear tool's profile**	**Approximation polynomial coefficients**
0	$x_A = X_A\cos\varphi_A - Y_A\sin\varphi_A + R_{rp}$ $\eta_A = X_A\sin\varphi_A + Y_A\cos\varphi_A + R_{rp}\cdot\varphi_A$	$D_x = x_A$ $D_\eta = \eta_A$
1/3	$x_B = X_B\cos\varphi_B - Y_B\sin\varphi_B + R_{rp}$ $\eta_B = X_B\sin\varphi_B + Y_B\cos\varphi_B + R_{rp}\cdot\varphi_B$	$B_x = \dfrac{18\cdot x_C - 9\cdot x_B + 2\cdot x_A - 5\cdot x_D}{6}$ $B_\eta = \dfrac{18\cdot \eta_C - 9\cdot \eta_B + 2\cdot \eta_A - 5\cdot \eta_D}{6}$ $C_x = \dfrac{-5\cdot x_A + 2\cdot x_D + 18\cdot x_B - 9\cdot x_C}{6}$ $C_\eta = \dfrac{-5\cdot \eta_A + 2\cdot \eta_D + 18\cdot \eta_B - 9\cdot \eta_C}{6}$
2/3	$x_C = X_C\cos\varphi_C - Y_C\sin\varphi_C + R_{rp}$ $\eta_C = X_C\sin\varphi_C + Y_C\cos\varphi_C + R_{rp}\cdot\varphi_C$	
1	$x_D = X_D\cos\varphi_D - Y_D\sin\varphi_D + R_{rp}$ $\eta_D = X_D\sin\varphi_D + Y_D\cos\varphi_D + R_{rp}\cdot\varphi_D$	$A_x = x_D$ $A_\eta = \eta_D$

2.1. Theoretical method and computational error

It is paramount to assess the profiling error of the rack-gear tool via the approximated method above by obtaining a solution using classical analytical methods as illustrated in [8].

For the profiles family described in (4), the enwrapping condition [1], [6] can be written as:

$$\frac{x'_u}{x'_j} = \frac{h'_u}{h'_j}, \tag{18}$$

where, x'_u, x'_j, h'_u, h'_j are partial derivatives of the profiles family equations (4) with respect to the variable parameters:

$$\begin{aligned} &x'_u = \cos(a + j);\\ &h'_u = \sin(a + j);\\ &x'_j = -\left(X_A + u\cdot\cos a\right)\cdot\sin j -\\ &-\left(Y_A + u\cdot\sin a\right)\cdot\cos j;\\ &h'_j = \left(X_A + u\cdot\cos a\right)\cdot\cos j -\\ &-\left(Y_A + u\cdot\sin a\right)\cdot\sin j + R_{rp}. \end{aligned} \tag{19}$$

Together with the family equations (4) and the enwrapping condition, one can define in the reference system xh, the rack-gear tool's profile reciprocally enwrapping with the profile to be generated (2).

Since the *u* parameter can be varied between the following limits:

$$u_{\min} = 0 \text{ and } u_{\max} = \sqrt{\left(X_B - X_A\right)^2 + \left(Y_B - Y_A\right)^2}, \tag{20}$$

the rack-gear profile can be expressed by the coordinates matrix (a large number of points on the active profile):

$$S = \begin{bmatrix} x_1 h_1 \\ \vdots \\ x_i h_i \\ \vdots \\ x_n h_n \end{bmatrix}. \tag{21}$$

The goal is to establish the maximum error value of the approximated profile at given points relative to the profile (21) obtained using classical algorithms.

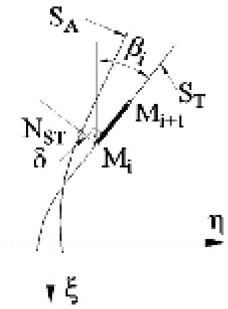

Fig. 2 – Relative position of theoretical —S_T and approximate —S_A profiles

For two successive points from the theoretical profile $M_i[x_i, h_i]$ and $M_{i+1}[x_{i+1}, h_{i+1}]$, the profile may be replaced with the $\overline{M_i M_{i+1}}$ segment expressed as:

$$\begin{aligned} x &= x_i - t\cdot\cos b_i;\\ h &= h_i + t\cdot\sin b_i, \end{aligned} \tag{22}$$

with

$$\tan b = \left|\frac{h_{i+1} - h_i}{x_{i+1} - x_i}\right| \tag{23}$$

and *t* a variable parameter.

The normal to the $[M_i, M_{i+1}]$ segment can be written as:

$$N_{ST}:\quad -\left(x - x_i\right)\cos b_i + \left(h - h_i\right)\sin b_i = 0. \tag{24}$$

The distance measured along the normal, at the point on the approximated profile (15), determine the error value of the *i* point on the analytical profile. In this way, if we label x_{A_i}, h_{A_i} the coordinates of intersection point between the normal $\overrightarrow{N_{ST}}$ and one of the approximation curves, the error value in the M_i point is defined as:

$$d = \sqrt{\left(x_{A_i} - x_i\right)^2 + \left(h_{A_i} - h_i\right)^2}. \tag{25}$$

By increasing the order of the Bezier approximation polynomial, the error d is expected to decrease. The numerical examples below demonstrate this observation. Moreover, the approximation polynomials of 2nd or 3rd degree lead to errors which are acceptable in most engineering practice.

3. Generation of a circular arc with a rack-gear tool

Consider next the case of generation of a circular arc profile using a rack-gear tool as shown in figure 3.

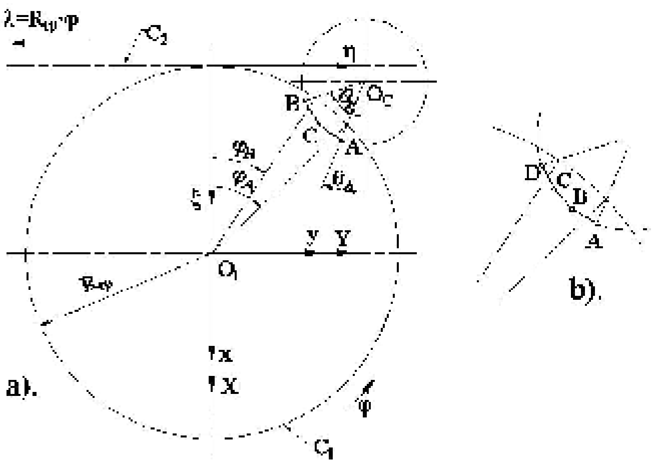

Fig. 3 – *Circular profile associated with the rolling centrodes couple: a). for 3 characteristic points; b). for 4 characteristic points; C_1 piece's centrode, C_2 tool's centrode.*

The circular arc is defined using the two end points, $A[X_A, Y_A]$ and $B[X_B, Y_B]$ and the radius *r*. The circle's center $O_C[X_C, Y_C]$ can then be easily determined.

The circle's parametric equations (the primary profile to be generated) can be written as:

$$C\left|\begin{array}{l} X(q) = X_{O_C} + r\cdot\cos q; \\ Y(q) = Y_{O_C} - r\cdot\sin q; \end{array}\right. \qquad (26)$$

with q variable parameter between the limits

$$q_A = \arccos\left(\frac{|X_A| - |X_{O_C}|}{r}\right)$$

and

$$q_B = \arccos\left(\frac{|X_B| - |X_{O_C}|}{r}\right). \qquad (27)$$

Based on equations (4), the circular's arc family in the rolling motion of the two centrodes can be written as:

$$(C)_j\left|\begin{array}{l} x = \left(X_{O_C} + r\cdot\cos q\right)\cos j - \\ \quad -\left(Y_{O_C} - r\cdot\sin q\right)\sin j + R_{rp}; \\ h = \left(X_{O_C} + r\cdot\cos q\right)\sin j + \\ \quad +\left(Y_{O_C} - r\cdot\sin q\right)\cos j + R_{rp}\cdot j. \end{array}\right. \qquad (28)$$

Similarly, the values of the angular parameter j in the rolling motion of the two centrodes can be determined from the normal to the circle at the current point,

$$N_C:\ [X - X(q)]\sin q + [Y - Y(q)]\cos q = 0 \qquad (29)$$

and the C_1 centrode equations:

$$C_1\left|\begin{array}{l} X = R_{rp}\cdot\cos j; \\ Y = R_{rp}\cdot\sin j. \end{array}\right. \qquad (30)$$

The angle j, corresponding to the characteristic points on the circle, is then determined as

$$j = \arcsin\left[\frac{X(q)\sin q + Y(q)\cos q}{R_{rp}}\right] - q. \qquad (31)$$

Equations (28) and (31) assembled together represent the rack-gear theoretical profile.

In tables 3 and 4, the enwrapping condition in (31) is illustrated for the case when second degree and a third degree polynomials are used, respectively.

Table 3. Circular arc, approximation polynomial of 2nd degree coefficients identification

θ	Primary profile	Enwrapping condition
θ_A	$X_A = X_{O_C} + r \cdot \cos q_A$ $Y_A = Y_{O_C} - r \cdot \sin q_A$	$j_A = \arcsin\left(\dfrac{X_A \sin q_A + Y_A \cos q_A}{R_{rp}}\right) - q_A$
θ_C	$q_C = q_A + (q_B - q_A)/2$ $X_A = X_{O_C} + r \cdot \cos q_C$ $Y_A = Y_{O_C} - r \cdot \sin q_C$	$j_C = \arcsin\left(\dfrac{X_C \sin q_C + Y_C \cos q_C}{R_{rp}}\right) - q_C$
θ_B	$X_B = X_{O_C} + r \cdot \cos q_B$ $Y_B = Y_{O_C} - r \cdot \sin q_B$	$j_B = \arcsin\left(\dfrac{X_B \sin q_B + Y_B \cos q_B}{R_{rp}}\right) - q_B$
λ	**Points on the rack-gear profile**	**Approximation polynomial coefficients**
1	$x_A = X_A \cos j_A - Y_A \sin j_A + R_{rp}$ $h_A = X_A \sin j_A + Y_A \cos j_A + R_{rp} \cdot j_A$	$A_x = x_A$ $A_h = h_A$
0.5	$x_C = X_C \cos j_C - Y_C \sin j_C + R_{rp}$ $h_C = X_C \sin j_C + Y_C \cos j_C + R_{rp} \cdot j_C$	$C_x = (x_C - 0.25 \cdot x_A - 0.25 \cdot x_B)/0.5$ $C_h = (h_C - 0.25 \cdot h_A - 0.25 \cdot h_B)/0.5$
0	$x_B = X_B \cos j_B - Y_B \sin j_B + R_{rp}$ $h_B = X_B \sin j_B + Y_B \cos j_B + R_{rp} \cdot j_B$	$B_x = x_B$ $B_h = h_B$

Table 4. Circle arc, approximation polynomial of 3rd degree coefficients identification

θ	Primary profile	Enwrapping condition
θ_A	$X_A = X_{O_C} + r \cdot \cos q_A$ $Y_A = Y_{O_C} - r \cdot \sin q_A$	$j_A = \arcsin\left(\dfrac{X_A \sin q_A + Y_A \cos q_A}{R_{rp}}\right) - q_A$
θ_B	$q_B = q_A + (q_D - q_A)/3$ $X_B = X_{O_C} + r \cdot \cos q_B$ $Y_B = Y_{O_C} - r \cdot \sin q_B$	$j_B = \arcsin\left(\dfrac{X_B \sin q_B + Y_B \cos q_B}{R_{rp}}\right) - q_B$
θ_C	$q_C = q_A + 2 \cdot (q_D - q_A)/3$ $X_C = X_{O_C} + r \cdot \cos q_C$ $Y_C = Y_{O_C} - r \cdot \sin q_C$	$j_C = \arcsin\left(\dfrac{X_C \sin q_C + Y_C \cos q_C}{R_{rp}}\right) - q_C$
θ_D	$X_D = X_{O_C} + r \cdot \cos q_D$ $Y_D = Y_{O_C} - r \cdot \sin q_D$	$j_D = \arcsin\left(\dfrac{X_D \sin q_D + Y_D \cos q_D}{R_{rp}}\right) - q_D$
λ	**Points on the rack-gear profile**	**Approximation polynomial coefficients**
0	$x_A = X_A \cos j_A - Y_A \sin j_A + R_{rp}$ $h_A = X_A \sin j_A + Y_A \cos j_A + R_{rp} \cdot j_A$	$D_x = x_A$ $D_h = h_A$
1/3	$x_B = X_B \cos j_B - Y_B \sin j_B + R_{rp}$ $h_B = X_B \sin j_B + Y_B \cos j_B + R_{rp} \cdot j_B$	$B_x = (18 \cdot x_C - 9 \cdot x_B + 2 \cdot x_A - 5 \cdot x_D)/6$ $B_h = (18 \cdot h_C - 9 \cdot h_B + 2 \cdot h_A - 5 \cdot h_D)/6$
2/3	$x_C = X_C \cos j_C - Y_C \sin j_C + R_{rp}$ $h_C = X_C \sin j_C + Y_C \cos j_C + R_{rp} \cdot j_C$	$C_x = (-5 \cdot x_A + 2 \cdot x_D + 18 \cdot x_B - 9 \cdot x_C)/6$ $C_h = (-5 \cdot h_A + 2 \cdot h_D + 18 \cdot h_B - 9 \cdot h_C)/6$
1	$x_D = X_D \cos j_D - Y_D \sin j_D + R_{rp}$ $h_D = X_D \sin j_D + Y_D \cos j_D + R_{rp} \cdot j_D$	$A_x = x_D$ $A_h = h_D$

4. Profiling an involute arc using a rack-gear tool

In figure 4 the coordinate systems associated with the workpiece and the tool for the manufacturing of an involute profile are shown. The parametrical equations of the involute with a radius circle R_b are:

$$E\left|\begin{array}{l} X(q) = -R_b \cdot \cos q - R_b \cdot q \cdot \sin q; \\ Y(q) = R_b \cdot \sin q - R_b \cdot q \cdot \cos q. \end{array}\right. \qquad (32)$$

The variation limits of the q parameter are established based on the internal (R_i) and the external (R_e) radii between which the profile extends:

$$q_A = \frac{\sqrt{R_i^{\,2} - R_b^{\,2}}}{R_b} \text{ and } q_B = \frac{\sqrt{R_e^{\,2} - R_b^{\,2}}}{R_b}. \qquad (33)$$

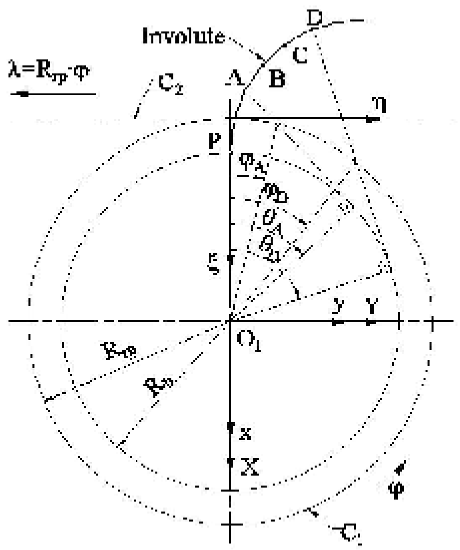

Fig. 4 – *Involute arc profile associated with the rolling centrodes couple*

From the condition that the normal at the involute,

$$N_E: \; [X - X(q)](-\cos q) + [Y - Y(q)]\sin q = 0, \qquad (34)$$

must intersect the rolling circle:

$$C_1\left|\begin{array}{l} X = -R_{rp} \cdot \cos j; \\ Y = R_{rp} \cdot \sin j, \end{array}\right. \qquad (35)$$

where j is a continuous variable parameter, one obtains the equation

$$j = \arccos\left(\frac{R_b}{R_{rp}}\right) + q, \qquad (36)$$

representing the Willis enwrapping condition [1]. Equations (4), (32) and (36) represent the rack-gear tool's theoretical profile.

When a third degree polynomial is used to approximate the tool's profile, the enwrapping condition is illustrated in table 5, for only 4 points on involute.

Table 5. Points belongs to the involute profile; enwrapping condition

θ	l	**Primary profile**	**Enwrapping condition**
θ_A	0	$X_A = -R_b \cdot \cos q_A - -R_b \cdot q_A \cdot \sin q_A$ $Y_A = R_b \cdot \sin q_A - -R_b \cdot q_A \cdot \cos q_A$	$j_A = \arccos\left(\frac{R_b}{R_{rp}}\right) + q_A$
θ_B	$\frac{1}{3}$	$q_B = q_A + \frac{q_D - q_A}{3}$ $X_B = -R_b \cdot \cos q_B - -R_b \cdot q_B \cdot \sin q_B$ $Y_B = R_b \cdot \sin q_B - -R_b \cdot q_B \cdot \cos q_B$	$j_B = \arccos\left(\frac{R_b}{R_{rp}}\right) + q_B$
θ_C	$\frac{2}{3}$	$q_C = q_A + 2\frac{q_D - q_A}{3}$ $X_C = -R_b \cdot \cos q_C - -R_b \cdot q_C \cdot \sin q_C$ $Y_C = R_b \cdot \sin q_C - -R_b \cdot q_C \cdot \cos q_C$	$j_C = \arccos\left(\frac{R_b}{R_{rp}}\right) + q_C$
θ_D	1	$X_D = X_{O_C} + +r \cdot \cos q_D$ $Y_D = Y_{O_C} - -r \cdot \sin q_D$	$j_D = \arccos\left(\frac{R_b}{R_{rp}}\right) + q_D$

Note: The calculation of points belonging to the approximated rack-gear profile is identical with the calculations presented in table 4.

5. Rack-gear profiling for a trochoid arc

5.1. Theoretical model

The trochoid described by a generic point M belonging to the circle r rolling on the circle with radius R has the following equations:

$$\Sigma \left| \begin{aligned} X &= r\cdot\cos(q+y)-(R+r)\cos y\,; \\ Y &= -r\cdot\sin(q+y)+(R+r)\sin y\,, \end{aligned} \right. \qquad (37)$$

where $$q=\frac{R}{r}y \qquad (38)$$

and y is a variable angular parameter.

From the intersection condition between the normal at the trochoidal curve Σ,

$$N_{\Sigma}\ [X-X(y)]X'_{y}+[Y-Y(y)]Y'_{y}=0\,, \qquad (39)$$

and the circle of radius $R_{rp}\equiv R$:

$$C_1 \left| \begin{aligned} X &= R_{rp}\cdot\cos j\,; \\ Y &= R_{rp}\cdot\sin j\,, \end{aligned} \right. \qquad (40)$$

one can obtain the theoretical enwrapping condition

$$j=y\,. \qquad (41)$$

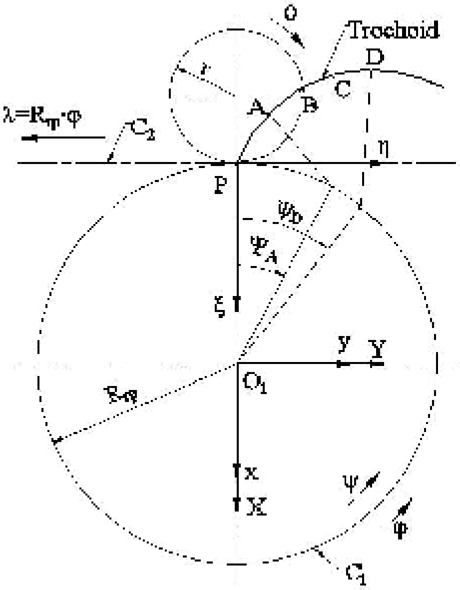

Fig. 5 – *Trochoidal profile associated with the rolling centrodes couple*

Equations (4), (37) and (41) represent the theoretical profile of the rack-gear reciprocally enveloping with the trochoidal profile.

For the arc $\overset{\frown}{AD}$ on the trochoidal curve, the enwrapping condition using approximation polynomials for the tool's profile is summarized in table 6, for 4 points known on the profile.

Table 6. Points belongs to the trochoid profile; enwrapping conditions

θ	l	**Primary profile**	**Enwrapping condition**
θ_A	0	$X_A=r\cdot\cos(q_A+y_A)-$ $-(R+r)\cdot\cos y_A$ $Y_A=-r\cdot\sin(q_A+y_A)+$ $+(R+r)\cdot\sin y_A$ $q_A=\frac{R}{r}y_A$	$j_A=y_A$
θ_B	$\frac{1}{3}$	$X_B=r\cdot\cos(q_B+y_B)-$ $-(R+r)\cdot\cos y_B$ $Y_B=-r\cdot\sin(q_B+y_B)+$ $+(R+r)\cdot\sin y_B$ $q_B=\frac{R}{r}y_B$ $y_B=y_A+\frac{\lvert y_D\rvert-\lvert y_A\rvert}{3}$	$j_B=y_B$
θ_C	$\frac{2}{3}$	$X_C=r\cdot\cos(q_C+y_C)-$ $-(R+r)\cdot\cos y_C$ $Y_C=-r\cdot\sin(q_C+y_C)+$ $+(R+r)\cdot\sin y_C$ $q_C=\frac{R}{r}y_C$ $y_C=y_A+2\frac{\lvert y_D\rvert-\lvert y_A\rvert}{3}$	$j_C=y_C$
θ_D	1	$X_D=r\cdot\cos(q_D+y_D)-$ $-(R+r)\cdot\cos y_D$ $Y_D=-r\cdot\sin(q_D+y_D)+$ $+(R+r)\cdot\sin y_D$ $q_D=\frac{R}{r}y_D$	$j_D=y_D$

Note: The calculation of points belonging to the approximated rack-gear profile is identical with the calculations presented in table 4.

6. Numerical examples

We start with a very simple rack-gear example to illustrate the methodology. In figure 6, a straight line profile to be generated is shown. The profile is determined by points $A[10;51.029]$; $B[10;59.160]$, associated with a circular centrode with radius $R_{rp} = 60$ mm.

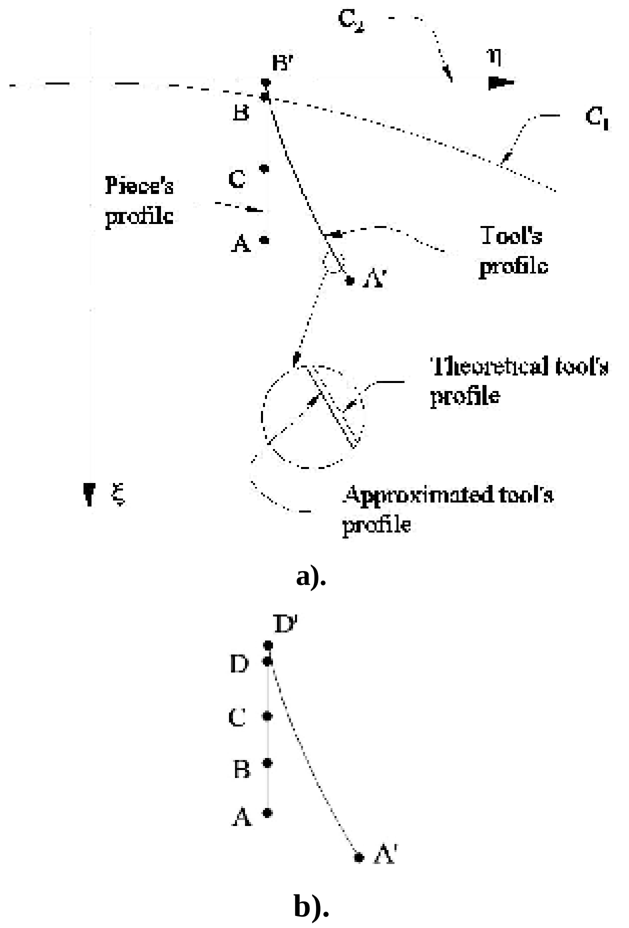

Fig. 6. *Linear profile example*

In the tables below, the numerical coordinates of the rack-gear profile obtained using a classical analytical method and the coordinates of the same tool obtained using the method outlined above are shown. In table 7, a 2nd order polynomial was used while in table 8 we show the results obtained using a 3rd order polynomial.

The error value defined according to equation (25) and also depicted in figure 6 is presented in these two tables. The maximum error value of the approximated profile with respect to the theoretical one is $e_{\max} = 0.018589$ mm for the approximation with a 2nd degree polynomial and $e_{\max} = 0.006$ mm for the approximation with a 3rd degree polynomial.

Table 7. Comparative results: approximated by 2nd degree polynomial and "theoretical" rack-gear profile for generation of the straight line profile

Approximated tool profile			**Theoretical tool profile**		**Error [mm]**
l	x **[mm]**	h **[mm]**	x **[mm]**	h **[mm]**	
0.0	**0.0008**	**10.0470**	**0.0008**	**10.0470**	**0.0000**
	0.4415	10.1417	0.4394	10.1300	0.0119
	0.9277	10.2564	0.9295	10.2397	0.0167
	⋮	⋮	⋮	⋮	⋮
	4.2687	11.3152	4.2757	11.3125	0.0075
	4.8623	11.5497	4.8612	11.5469	0.0031
0.5	**5.4496**	**11.7946**	**5.4496**	**11.7946**	**0.0000**
	6.0417	12.0541	6.0399	12.0551	0.0020
	6.6270	12.3229	6.6313	12.3279	0.0066
	⋮	⋮	⋮	⋮	⋮
	10.1751	14.1963	10.1721	14.1953	0.0032
	10.7558	14.5404	10.7574	14.5416	0.0020
1.0	**11.3405**	**14.8970**	**11.3405**	**14.8970**	**0.0000**

Table 8. Comparative results: approximated by 3rd degree polynomial and "theoretical" rack-gear profile for generation of the straight lined profile

Approximated tool profile			**Theoretical tool profile**		**Error [mm]**
l	x **[mm]**	h **[mm]**	x **[mm]**	h **[mm]**	
0.0	**0.0008**	**10.0470**	**0.0008**	**10.0470**	**0.0000**
	0.4436	10.1314	0.4394	10.1300	0.0045
	0.9316	10.2400	0.9295	10.2397	0.0022
	⋮	⋮	⋮	⋮	⋮
	2.5489	10.6963	2.5507	10.6980	0.0024
	3.1146	10.8853	3.1186	10.8872	0.0044
0.333	**3.4978**	**11.0208**	**3.4978**	**11.0208**	**0.0000**
	3.6911	11.0914	3.6943	11.0923	0.0033
	4.2768	11.3140	4.2757	11.3125	0.0019
	⋮	⋮	⋮	⋮	⋮
	6.6353	12.3310	6.6313	12.3279	0.0050
	7.2211	12.6117	7.2232	12.6123	0.0022
0.666	**7.4125**	**12.7058**	**7.4125**	**12.7058**	**0.0000**
	7.8194	12.9094	7.8149	12.9080	0.0047
	8.4048	13.2117	8.4060	13.2145	0.0031
	⋮	⋮	⋮	⋮	⋮
	10.1775	14.1942	10.1721	14.1953	0.0055
	10.7573	14.5385	10.7574	14.5416	0.0031
1.0	**11.3405**	**14.8970**	**11.3405**	**14.8970**	**0.0000**

Since in all cases the error is less than 0.01 mm, one can conclude that this new method is accurate enough.

Consider now the case of a rack-gear tool to generate a circular arc $\overset{\frown}{AD}$ ($\overset{\frown}{AB}$ for 3 points approximation) with radius $r = 25$ mm and ends points at $A[\text{-90; 1.5}]$, $\text{D}[\text{-75; 15}]$ for the 4 points approximation and $A[\text{-90; 1.5}]$, $\text{B}[\text{-75; 15}]$ for the 3 points approximation. In figure 7 and tables 9 and 10, we show the shapes and coordinates of rack-gear tool's profile reciprocally enveloping with a circular arc associated with a circular centrode with radius $R_{rp} = 81.22$ mm.

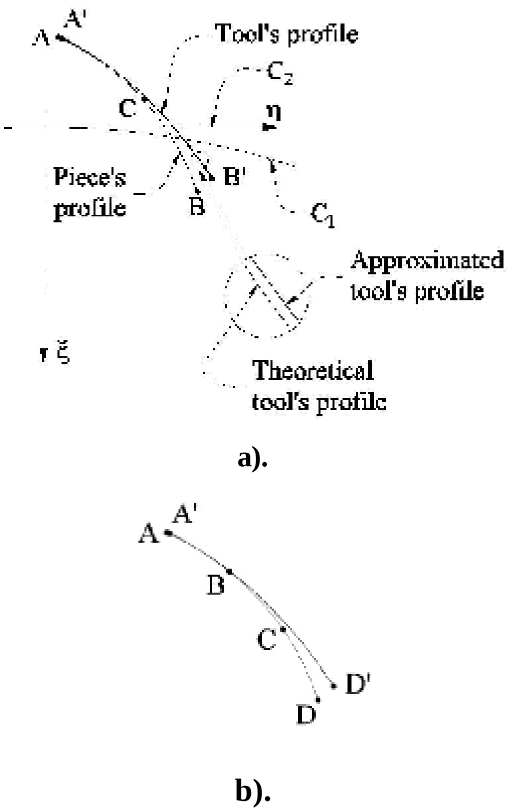

Fig. 7. *Circular arc profile*

Table 9. Comparative results: approximation by 2nd degree polynomial and "theoretical" rack-gear profile for generation of the circular arc profile

	Approximated tool profile		Theoretical tool profile		Error [mm]
l	x **[mm]**	h **[mm]**	x **[mm]**	h **[mm]**	
0.0	**-8.6920**	**1.7675**	**-8.6920**	**1.7675**	**0.0000**
	-8.2564	2.6204	-8.2625	2.6146	0.0084
	-7.7998	3.4473	-7.8073	3.4447	0.0080
	M	M	M	M	M
	-4.5254	8.1176	-4.5350	8.1187	0.0098
	-3.8950	8.8490	-3.9044	8.8454	0.0101
0.5	**-3.2514**	**9.5568**	**-3.2514**	**9.5568**	**0.0000**
	-2.5821	10.2562	-2.5741	10.2560	0.0080
	-1.8871	10.9471	-1.8711	10.9448	0.0161
	M	M	M	M	M
	2.9862	15.0334	2.9987	14.9979	0.0376
	3.9409	15.7154	3.9623	15.7015	0.0255
1.0	**4.9835**	**16.4260**	**4.9835**	**16.4260**	**0.0000**

Table 10. Comparative results: approximated by 3rd degree polynomial and "theoretical" rack-gear profile for generation of the circle arc profile

	Approximated tool profile		Theoretical tool profile		Error [mm]
l	x **[mm]**	h **[mm]**	x **[mm]**	h **[mm]**	
0.0	**-8.6920**	**1.7675**	**-8.6920**	**1.7675**	**0.0000**
	-8.2537	2.6154	-8.2625	2.6146	0.0088
	-7.7983	3.4402	-7.8073	3.4447	0.0101
	M	M	M	M	M
	-5.7268	6.6157	-5.7236	6.6227	0.0078
0.333	**-5.3406**	**7.1248**	**-5.3406**	**7.1248**	**0.0000**
	-5.1433	7.3769	-5.1404	7.3794	0.0038
	M	M	M	M	M
	-1.8709	10.9466	-1.8711	10.9448	0.0018
	-1.1463	11.6208	-1.1423	11.6241	0.0052
0.666	**-0.9032**	**11.8401**	**-0.9032**	**11.8401**	**0.0000**
	-0.3912	12.2914	-0.3850	12.2972	0.0085
	0.3956	12.9600	0.4034	12.9668	0.0104
	M	M	M	M	M
	2.9940	14.9969	2.9987	14.9979	0.0048
	3.9586	15.7011	3.9623	15.7015	0.0038
1.0	**4.9835**	**16.4260**	**4.9835**	**16.4260**	**0.0000**

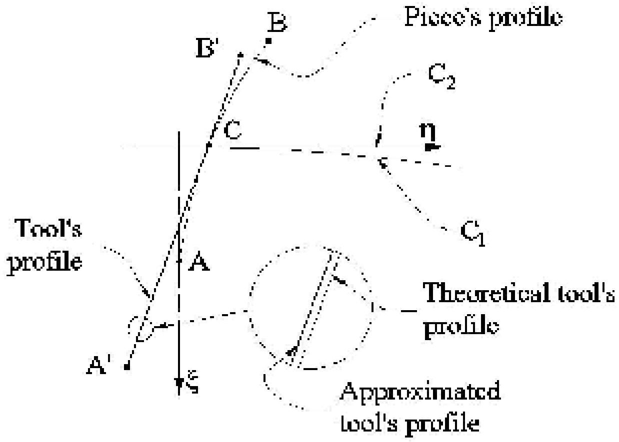

Fig. 8. *Involute arc profile*

The approximated profiles were determined based on the specified methodology for approximating polynomials of 2nd and 3rd degree. The maximum error for a 3rd degree approximation polynomial is smaller then 0.0137 mm. The two profiles (theoretical and approximated) are quite close to each other.

Last example considers the case of generating an involute arc with a rack-gear tool. In table 11, we show the rack-gear tool's profile determined by an analytical method and by approximation with a 3rd degree polynomial for the generation of an involute arc for $A[0; -164.44]$, $R_b = 164.44$ mm, $R_e = 185$ mm and $R_{rp} = 175$ mm (the teethed wheel profile for *m=5* mm and *z= 35* teeth). The maximum error of the approximated profile regarding the theoretical one is $err_{MAX} = 0.016425$ mm.

Table 11. Rack-gear profile; comparative result approximated by 3rd degree polynomial and theoretical involute arc profile

Approximated tool profile			**Theoretical tool profile**		**Error [mm]**
l	x **[mm]**	h **[mm]**	x **[mm]**	h **[mm]**	
0.0	**20.3656**	**-4.8042**	**20.3656**	**-4.8042**	**0.0000**
	14.0151	-2.4920	14.0155	-2.4922	0.0004
	11.3505	-1.5219	11.3363	-1.5167	0.0151
	M	M	M	M	M
	2.1557	1.8258	2.1514	1.8274	0.0046
	1.0261	2.2371	1.0387	2.2325	0.0133
0.5	**-0.0129**	**2.6154**	**-0.0129**	**2.6154**	**0.0000**
	-1.0207	2.9823	-1.0132	2.9796	0.0080
	-1.9668	3.3268	-1.9698	3.3279	0.0032
	M	M	M	M	M
	-7.0195	5.1664	-7.0132	5.1642	0.0066
	-7.7641	5.4375	-7.7596	5.4359	0.0048
1.0	**-8.4871**	**5.7007**	**-8.4871**	**5.7007**	**0.0000**

The maximum error of approximated profile corresponds to a high quality precision of the gear.

7. Conclusions

In this paper we have developed a methodology to represent the enwrapping profiles determined according to the fundamental laws of surfaces enveloping by using an approximate description of the enveloping profile via Bezier polynomials. The method was applied for the generation of surfaces associated with a couple of rolling centrodes (such as a rack-gear generating tool) starting from a very limited number of points along the profile to be generated.

Specific examples were presented for elementary profiles such as straight line segments, circular arcs and involute arc which can be used as building blocks for complex profiles of "teethed" pieces used in technical applications such as large side mills, gears, and splined shafts. The numerical examples show that the errors of the approximated profiles are small enough to be acceptable. In general, if the degree of the approximating Bezier polynomial increases, the profile approximation precision will increase.

The proposed method is quite economical. Using a small number of points on the workpiece profile (3 or 4 points) leads to approximation errors that are within engineering tolerances for most practical applications.

The method may be applied when points on the profile to be generated are known using 3D measuring machines.

The proposed method can be applied for others cases of generation by enveloping such as generation with gear-shape cutter tool or with rotary cutter tool.

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